Birzeit University
Faculty of Engineering and Technology
Department of Electrical and Computer Engineering
Information and Coding Theory ENEE 5304
Midterm Exam

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Problem 1: 22 Points

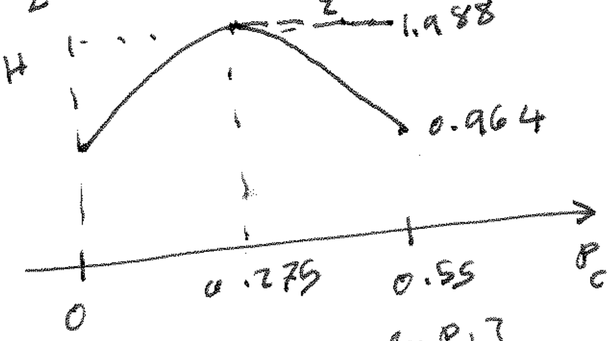
A discrete memory-less source produces one of four possible symbols every time unit with the following probabilities:

Symbol	A	B	C	D
Probability	0.25	0.2	P_C	P_D

9 a. Find values of P_C and P_D that minimize the source entropy.
7 b. Suppose that $P_C = 0.25$, find the amount of information, in bits, in the sequence (ABBD).
7 c. Again, suppose that $P_C = 0.25$, is it possible to find a prefix-free code with the following code-word lengths (1, 2, 2, 3) bits? Explain.

$P_D = 1 - (0.25 + 0.2 + P_c) = 0.55 - P_c$

a. $H = \sum -P_i \log_2 P_i = -0.25 \log_2 0.25 - 0.2 \log_2 0.2$
$- P_c \log_2 P_c - [0.55 - P_c] \log_2 [0.55 - P_c]$



H is min when
$P_c = 0,\ P_D = 0.55$
$P_c = 0.55,\ P_D = 0$

b. $P_a = 0.25 \quad P_b = 0.2$
$P_c = 0.25 \quad P_d = 0.3$

$I = -\log_2 P(ABBD) = -\frac{1}{\ln 2}[\ln P_a + \ln P_b + \ln P_b + \ln P_d]$
$= -\frac{1}{\ln 2}[\ln 0.25 + \ln 0.2 + \ln 0.2 + \ln 0.3]$
$= 8.380$ bits

c. Kraft's inequality
For a prefix-free code, one should have
$\sum 2^{-l_i} \leq 1$

$2^{-1} + 2^{-2} + 2^{-2} + 2^{-3} = \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} > 1$
$\Rightarrow$ It is not possible.

Problem 2: 18 Points

X is a random variable, which assumes the values 1 and -1 with probabilities

$$P(X = +1) = 0.15; \quad P(X = -1) = 0.85.$$

Y is another random variable, independent of X, which assumes the values 1 and -1 with probabilities

$$P(Y = +1) = 0.15; \quad P(Y = -1) = 0.85.$$

a. Find the entropy of X
b. Define Z = X + Y. Find the entropy of Z.

a. $H(X) = -0.15 \log_2 0.15 - 0.85 \log_2 0.85$

$H(X) = 0.609$

b. $Z = X + Y$

Z assumes the values

$Z = +2$ when $X = 1$ and $Y = +1 \Rightarrow P(X=1 \wedge Y=1) = P(X=1)\,P(Y=1)$
$= (0.15)(0.15) = 0.0225$

$Z = -2$ when $X = -1$ and $Y = -1 \Rightarrow P(X=-1 \wedge Y=-1) = P(X=-1)\,P(Y=-1)$
$= (0.85)(0.85) = 0.7225$

$Z = 0$ when $X = +1,\ Y = -1$; $X = -1,\ Y = +1$ $\Rightarrow P(Z=0) = 1 - (0.0225 + 0.7225$

$= 0.255$

Z	-2	0	2
P(Z)	0.7225	0.255	0.0225

$H(Z) = -0.7225 \log_2 0.7225 - 0.255 \log_2 0.255 - 0.0225 \log_2 0.0225$

$H(Z) = 0.9644$

Problem 3: 20 Points

A discrete memory-less source produces symbols A and B with probabilities P(A) = 3/4, P(B) = 1/4. The symbols are grouped into messages of size 2 symbols and applied to a Huffman encoder.

Use Huffman's algorithm to find the code-words which are assigned to the messages AA, AB, BA, BB.

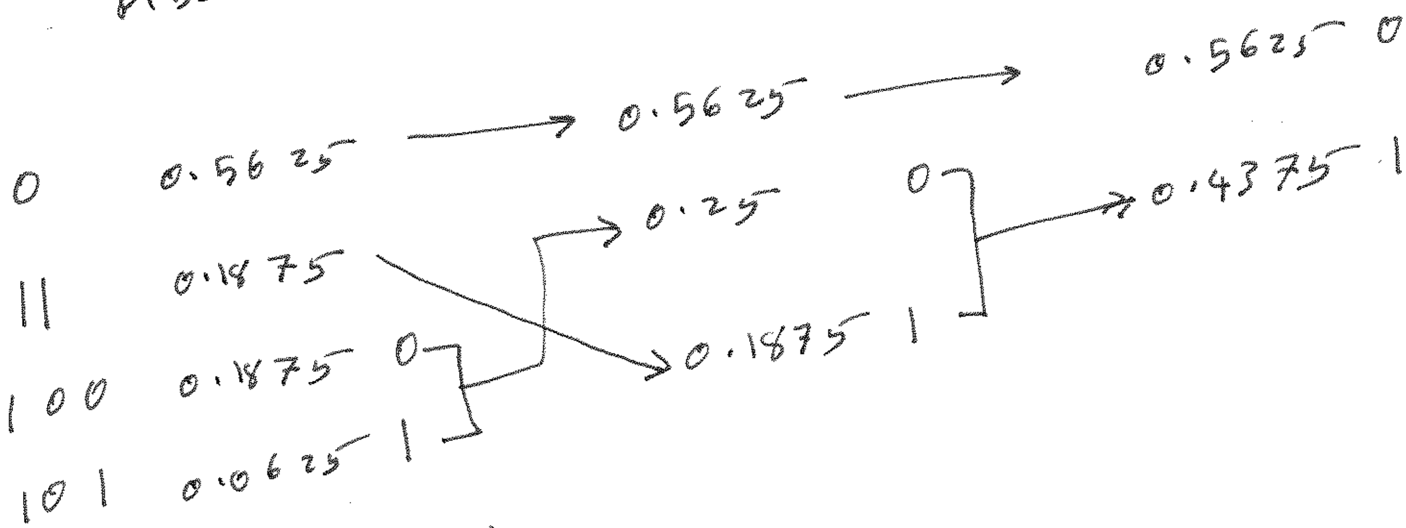
$P(A) = 0.75$ / $P(B) = 0.25$

$P(AA) = 0.75 \times 0.75 = 0.5625$

$P(AB) = 0.75 \times 0.25 = 0.1875$

$P(BA) = 0.25 \times 0.75 = 0.1875$

$P(BB) = 0.25 \times 0.25 = 0.0625$



0	AA
11	AB
100	BA
101	BB

Problem 4: 20 Points

A discrete memory-less source produces one of five possible symbols every time unit with probabilities given in the table below. Also, given in the table are code-words for the first three symbols and the lengths in bits for all five symbols.

Symbol	A	B	C	D	E
Probability	0.3	0.25	0.2	0.15	0.1
Code-word	00	01	11	C_D	C_E
Number of bits	2	2	2	3	3

a. Find the ratio of the source entropy to the average number of bits/source code-word.

b. Find two possible code-words C_D and C_E (each with three digits) such that the code is prefix-free.

$H = -\sum p_i \log_2 p_i$ 5.5

$H = 2.228$

$\bar{L} = 0.3(2) + 0.25(2) + 0.2(2) + 0.15(3) + 0.1(3)$

$= 2.25$ 5.5

$\Rightarrow \frac{H}{\bar{L}} = \frac{2.228}{2.25} = 0.9903$

b. $C_D = 100$ 4

$C_E = 101$ 4

Problem 5: 20 Points
The input X and the output Y of a discrete memoryless channel are related through the following joint probability mass function (both X and Y take on the values 0, 1, 2)

	X \ Y	0	1	2
		0.5	0.25	0.25
0.5	0	0.25	0.125	0.125
0.25	1	0.125	0.0625	0.0625
0.25	2	0.125	0.0625	0.0625

6 a. Find the entropies $H(X)$ and $H(Y)$.
8 b. Find the joint entropy $H(X;Y)$
6 c. Find P(Y = 0/X = 0). This is the transition probability that Y = 0 given X = 0.

a. $H(X) = -0.5 \log_2 0.5 - 2 * 0.25 \log_2 0.25$
$= 1.5$ bits/symbol

$H(Y) = 1.5$ bits/symbol

b. $H(X,Y) = -0.25 \log_2 0.25 - 4 * 0.125 \log_2 0.125 + 4 * 0.0625 \log_2 0.0625$

$= 3$ bits

c. $p(Y=0/X=0) = \dfrac{p(X=0 \cap Y=0)}{p(X=0)}$

$= \dfrac{0.25}{0.5}$

$= \dfrac{1}{2}$